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A Hybrid Differential Transforms and Finite Difference Method to Numerical Solution of Convection–Diffusion Equation

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ABSTRACT

In this work, we discuss a hybrid-based method on differential transforms and a finite difference method to numerical solution of convection–diffusion equation with Dirichlet's type boundary conditions. The developed method is tested on various problems and the numerical results are reported in tabular and figure form. This method can be easily extended to handle non-linear convection–diffusion partial differential equations.

Keywords- Hybrid method; Differential transform; Finite difference method; Convection–Diffusion equation.

I. INTRODUCTION

The term convection means the movement of molecules within fluids, whereas, diffusion describes the spread of particles through random motion from regions of higher concentration to regions of lower concentration [1, 2, 3]. Convection-diffusion equations model a variety of physical phenomena [4, 5]. The numerical solution of convection-diffusion transport problems arises in many important applications in science and engineering. Characteristic examples are the heat transfer through a permeable medium, the transport of a pollutant through the atmosphere or the transport of a fluid through the porous medium [6, 7, 8].

In this paper, the numerical solution of the convection-diffusion equation is examined using a hybrid differential transforms approach and a finite difference method, as proposed by Chu and Chen [9]. We have an equation with free variables and in terms of the free variable, we begin with the hydride differential transforms technique. Then, in terms of obtaining a numerical scheme to obtain the coefficients associated to the differential transform's method [10], we utilize the finite difference approach to estimate the function at specified node points. We avoid the computational complexity of the two-variable differential transform method by employing this method instead of the two-variable differential transform method [11]. We will also benefit from the differential transform method's computational accuracy [12]. Consider the equation for convection-diffusion [13]

$$u_t(x,t) + \varepsilon u_x(x,t) = \gamma u_{xx}(x,t), 0 < x < 1\,, \qquad t > 0 \quad (1)$$

The initial condition is

$$u(x,0) = \varphi(x), \quad 0 < x < 1 \qquad (2)$$

The following are the boundary conditions:

$$u(0,t) = g_0(t), \quad t \geq 0 \qquad (3)$$

$$u(1,t) = g_1(t), \quad t \geq 0 \qquad (4)$$

Where the parameters $\varepsilon, \gamma > 0$ are the viscosity coefficient and phase speed respectively and subscripts t and t denote differentiation. And g_0, g_1 and φ are known functions with sufficient smoothness.

II. METHOD OF HYBRID DIFFERENTIAL TRANSFORM

We can approximate $u(x,t)$ for the free variable t using the hybrid differential transform as follows

$$u(x,t) \approx \sum_{k=0}^{k} Y_{\omega,k} t^k \qquad (5)$$

Consequently, we have

$$u_t(x,t) \approx \sum_{k=0}^{k} (k+1) Y_{\omega,k+1} t^k \qquad (6)$$

In addition, we can deduce from equation (2) that:

$$u_x(x,t) \approx \sum_{k=0}^{k} \frac{dY_{\omega,k}}{dx} t^k \qquad (7)$$

$$u_{xx}(x,t) \approx \sum_{k=0}^{k} \frac{d^2 Y_{\omega,k}}{dx^2} t^k \qquad (8)$$

As a result of the differential translation of equation (4) with respect to the intendant variable t, we get

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$$(k+1)Y_{x,k+1} + \varepsilon \frac{dY_{x,k}}{dx} = \gamma \frac{d^2Y_{x,k}}{dx^2}, \quad k = 0,1,\dots,K-1 \qquad (9)$$

Using the finite difference approach to discretize equation (9) we divide the interval $[0,1]$ by the step size h as $x\_i = ih$, $i = 0,1,\dots,N$ and $N = 1/h$ as follows

$$(k+1)Y_{x_i,k+1} + \varepsilon \frac{Y_{x_{i+1},k} - Y_{x_{i-1},k}}{2h} = \gamma \frac{Y_{x_{i+1},k} - 2Y_{x_i,k} + Y_{x_{i-1},k}}{h^2}, k = 0,1,\dots,K-1, i = 1,2,\dots,N-1 \qquad (10)$$

We have simplified $Y\_(x\_i,k) = Y\_(i(k))$ by introducing the symbolization $Y\_(x\_i,k) = Y\_(i(k))$.

$$Y_{k+1}^i = \frac{1}{k+1}\left(\frac{\gamma}{h^2} Y_k^{i+1} - 2Y_k^i + Y_k^{i-1} - \frac{\varepsilon}{2h} Y_k^{i+1} - Y_k^{i-1}\right),$$
$$k = 0,1,\dots,K-1,\ i = 1,2,\dots,N-1 \qquad (11)$$

To use scheme (11), we require the values $Y\_k\^0$ and $Y\_k\^N$ for $k = 0,1,\dots,K$ and $Y\_0\^i$ for $i = 0,1,\dots,N$. To find the values of $Y\_0\^i$, we put $t = 0$ in equation (5), therefore we have according to the initial condition (2).

$$\varphi(x_i) = \sum_{k=0}^{K} Y_k^i$$

Consequently,

$$Y_0^i = \varphi(x_i), \quad i = 0,1,\dots,N \qquad (12)$$

We used $x = 0$ in equation (5) to find the values of $Y\_k\^0$, so according to the boundary condition (3), we have

$$g_0(t) = \sum_{k=0}^{K} Y_k^0 t^k \qquad (13)$$

Using equation (13), create a system of linear equations by determining various values for \, such as $t = 0,1,\dots,K$.

$$AY_0 = b_0 \qquad (14)$$

We reach to the point where.

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^k \\ \vdots & \vdots & \vdots & & \\ 1 & K & K^2 & \dots & K^k \end{pmatrix}, Y_0 = \begin{pmatrix} Y_0(1) \\ Y_0(2) \\ Y_0(3) \\ \vdots \\ Y_0(K) \end{pmatrix}, b_0 = \begin{pmatrix} g_0(1) \\ g_0(2) \\ g_0(3) \\ \vdots \\ g_0(K) \end{pmatrix}$$

The values $Y\_k\^0$ for $k = 0,1,\dots,K$ are obtained from the solution of system (14). In the same way, the boundary condition yields the following system of linear equations (4).

$$AY_N = b_1 \qquad (15)$$

Where,

$$Y_N = \begin{pmatrix} Y_N(1) \\ Y_N(2) \\ Y_N(3) \\ \vdots \\ Y_N(K) \end{pmatrix}, \qquad b_1 = \begin{pmatrix} g_1(1) \\ g_1(2) \\ g_1(3) \\ \vdots \\ g_1(K) \end{pmatrix}.$$

The values of $Y\_k\^N$ for $k = 0,1,\dots,K$ will be obtained from the solution of system (15).

As a result, of $Y\_k\^i$ for $k = 0,1,\dots,K$ and $i = 0,1,\dots,N$ can be calculated using equation (11), as well as equations (12), (14) and (15).

Now we let T be the last time. Divide the interval $[0,T]$ by the step size τ as $t\_j = j\tau$, where $j = 0,1,\dots,M$ and $M = T/\tau$ .Let $u(x\_i,t\_j) \approx U\_i\^j$ be the approximate solution value. After calculating $Y\_k\^i$ and using equation (5), we get.

$$U_i^j = \sum_{k=0}^{K} Y_k^i t_j^k \qquad (16)$$

III. NUMERICAL EXPERIMENTS

Example 1: In equation (1), we assume $\varepsilon = 0\_\cdot 1$ and $\gamma = 00\_\cdot 1$; in this case, we have $\varphi(x) = e\^\alpha x, g\_0\ (t) = e\^\beta t$, and$g_1\ (t) = e\^(\alpha + \beta t$, where $\alpha = 1\_\cdot 17712434446770$, $\beta = -0\_\cdot 09,$ and the exact solution of the equation is $u(x,t) = e\^(\alpha + \beta t)$. In Table 1, the values of the exact solution, the approximate solution, and the absolute error at the final time $T = 0{,}5$ are given for different x_i, assuming $h = 0.1$, $\tau = 0.01$, $k = 20$, and $T = 0{,}5$. Figures 1 and 2 show the exact and approximate solution diagrams at all points of the node.

Example 2: In this example let $\varepsilon = 3.5$, $\gamma = 0.022, \alpha = 0.028547, \beta = -0.0999, \tau = 0.01, K = 20, and\ T = 0.5$. The values of the exact solution, the approximate solution, and the absolute error at the final time $t = 0.5$ are given in Table 2 for various x_i . Figures 3 and 4 show the exact and approximate solution diagrams at all points of the node.

Example 3: In equation (1-4), we let $\varepsilon = 0.8$ and $\gamma = 0.1$ be positive numbers. The exact solution of the equation will be given in this case.

$$u(x,t) = \sqrt{\frac{20}{20+t}}\, e^{-\frac{(x-2+\varepsilon t)^2}{4\gamma(t+20)}}$$

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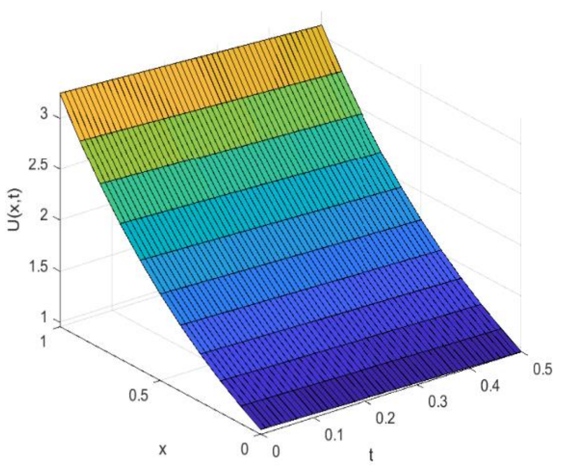
In addition, the initial and boundary conditions are obtained using the exact solution. We also make the assumption.

$$h = 0.1, \tau = 0.01, \quad K = 10, \text{and } T = 0.1$$

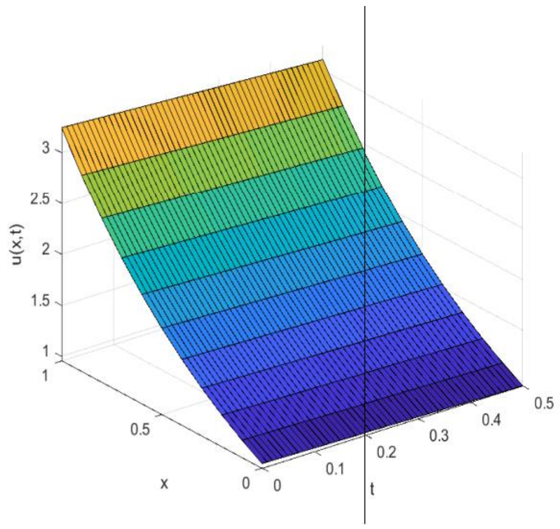
For different x_i , the values of exact solution, approximate solution, and absolute error at the final time $t = 0.01$ are given in Table 3, and the exact and approximate solution diagrams at all node points are shown in Figures 5 and 6, respectively.

Table 1: The absolute error, the relative error and the exact and approximate solution for Example 1 at time $t = 0.5$.

x	*Exact solution*	*Hybrid method*	*Absolute error*	*Relative error*
0.1	1.075421	1.075335	$8.59385e-05$	$7.9911e-05$
0.2	1.209763	1.209632	$1.31971e-04$	$1.0909e-04$
0.3	1.360888	1.360728	$1.59944e-04$	$1.1753e-04$
0.4	1.530891	1.530708	$1.82977e-04$	$1.1952e-04$
0.5	1.722130	1.721924	$2.06486e-04$	$1.1990e-04$
0.6	1.937260	1.937028	$2.32144e-04$	$1.1983e-04$
0.7	2.179264	2.179004	$2.59152e-04$	$1.1892e-04$
0.8	2.451499	2.451220	$2.78759e-04$	$1.1371e-04$
0.9	2.757741	2.757494	$2.46989e-04$	$8.9562e-05$



Figures 1: The exact solution diagram for Example 1.



Figures 2: The numerical solution diagram for Example 1.

Table 2: The absolute error, the relative error and the exact and approximate solution for Example 2 at time $t = 0.5$.

x	*Exact solution*	*Hybrid method*	*Absolute error*	*Relative error*
0.1	0.9539966	0.9540015	4.93028e-06	5.1680e-06
0.2	0.95677239	0.9566050	1.18955e-04	1.2434e-04
0.3	0.9594591	0.9594491	9.94557e-06	1.0366e-05
0.4	0.9622021	0.9424171	2.15085e-04	2.2353e-04
0.5	0.9649529	0.9649515	1.34050e-06	1.3892e-06
0.6	0.9677116	0.9674866	2.24950e-04	2.3246e-04
0.7	0.9704781	0.9704970	1.887800-05	1.9452e-05
0.8	0.97322526	0.9733982	1.45559e-04	1.4956e-04
0.9	0.9706350	0.9760204	1.46422e-05	1.5002e-05

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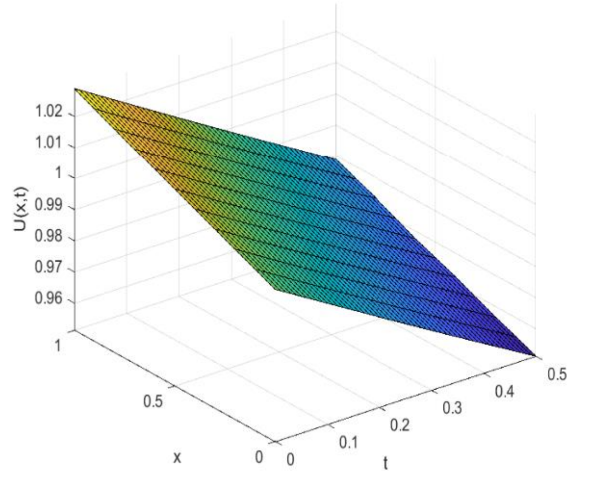
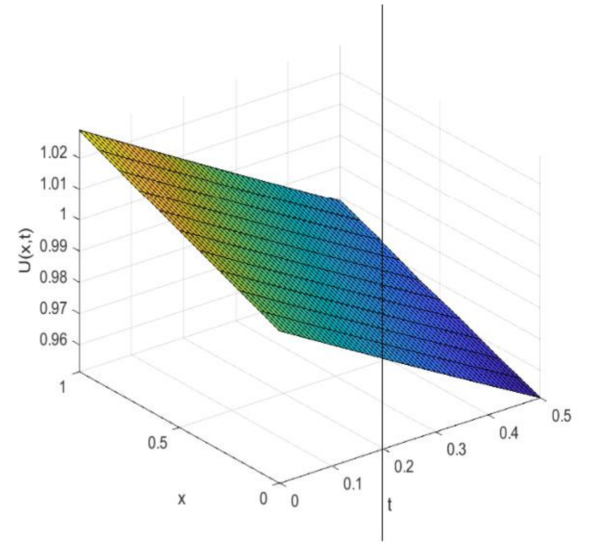


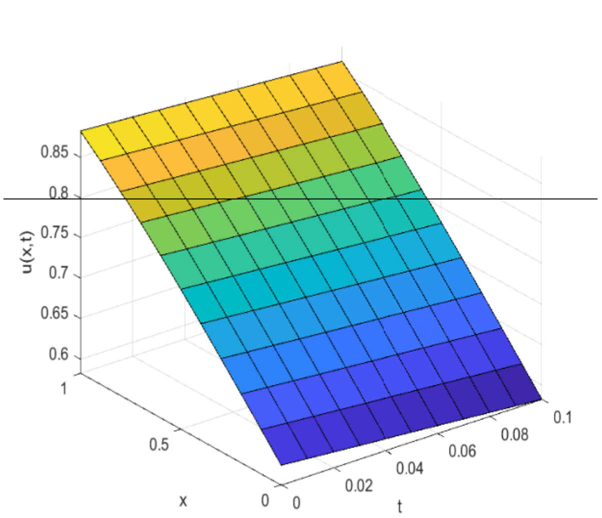
Figures 3: The exact solution diagram for Example 2.



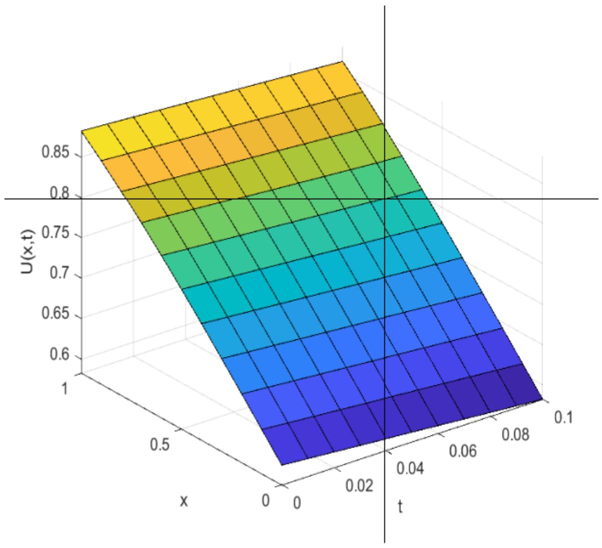
Figures 4: The numerical solution diagram for Example 2.

Table 3: The absolute error, the relative error and the exact and approximate solution for Example 3 at time $t = 0.5$.

x	*Exact solution*	*Hybrid method*	*Absolute error*	*Relative error*
0.1	0.6125616	0.6125786	$1.69545e-05$	$2.7678e-05$
0.2	0.6426881	0.6427079	$1.98710e-05$	$3.0779e-05$
0.3	0.6726209	0.6726421	$2.12228e-05$	$3.1552e-05$
0.4	0.7021989	0.7022210	$2.20454e-05$	$3.1395e-05$
0.5	0.7312563	0.7312788	$2.25249e-05$	$3.0803e-05$
0.6	0.7596241	0.7596469	$2.27565e-05$	$2.9958e-05$
0.7	0.7871319	0.7871545	$2.26034e-05$	$2.8716e-05$
0.8	0.8136094	0.8136306	$2.11635e-05$	$2.6012e-05$
0.9	0.8388882	0.839009	$1.269140-05$	$1.5129e-05$



Figures 5: The exact solution diagram for Example 3.



Figures 6: The numerical solution diagram for Example 3.

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IV. CONCLUSION

In this work the hybrid differential and finite difference method is used to numerical solution of convection-diffusion equations. Three numerical examples are illustrated in tables and figures. The hybrid method provides an iterative procedure to calculate the numerical solutions; therefore, it is not necessary to carry out complicated symbolic computation. Moreover, the hybrid method provides an iterative procedure to calculate the numerical solutions without using linearization. Comparisons of the results with exact solutions showed that the present method is capable of solving the given equations very well.

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