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Finite Integral Involving the Generalized Modified I-Function of Two Variables

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ABSTRACT

In the present paper, we evaluate the general finite integral involving the generalized modified I-functions of two variables. At the end, we shall see several corollaries and remarks.

Keywords- Modified generalized I-function of two variables, generalized I-function of two variables, generalized modified H-function of two variables, generalized modified Meijer-function of two variables, I-function of two variables, H-function of two variables, Meijer-function of two variables, double Mellin-Barnes integrals contour, finite integral.

2010 Mathematics Subject Classification- 33C05, 33C60.

I. INTRODUCTION AND PRELIMINARIES

Kumari et al. [8] have defined the I-function of two variables. On the other hand, Prasad and Prasad [10] have studied the modified generalized H-function of two variables. Recently Singh and Kumar [15] have worked about the modified of generalized I-function of two variables. This function is an extension of the I-function of two variables and modified of generalized H-function of two variables at the time. In this paper, first, we define the modified generalized I-function of two variables. Then we calculate the generalized finite integral involving this function. At the last section, we will see several cases and remarks. The double Mellin-Barnes integrals contour occurring in this paper will be referred to as the generalized modified I-function of two variables throughout our present study and will be defined and represented as follows

We have

$$I(z_1,z_2)=I^{m_1,n_1,m_2,n_2,m_3,n_3,m_4,n_4}_{p_1,q_1,p_2,q_2;p_3,q_3;p_4,q_4}\left(\begin{array}{c|c} z_1 & \{(\mathrm{a}_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1}:\{(c_j;\gamma_j,C_j;\mathbf{C}_j)\}_{1,p_2}:\{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3},\{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ z_2 & \{(\mathrm{b}_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1}:\{(d_j;\delta_j,D_j;\mathbf{D}_j)\}_{1,q_2}:\{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3},\{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4} \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)z_1^s z_2^t\mathrm{d}s\mathrm{d}t \tag{1.1}$$

where

$$\theta(s,t)=\frac{\prod_{j=1}^{m_1}\Gamma^{\mathbf{B}_j}(b_j-\beta_j s-B_j t)\prod_{j=1}^{n_1}\Gamma^{\mathbf{A}_j}(1-a_j+\alpha_j s+A_j t)\prod_{j=1}^{m_2}\Gamma^{\mathbf{D}_j}(d_j-\delta_j s+D_j t)}{\prod_{j=m_1+1}^{q_1}\Gamma^{\mathbf{B}_j}(1-b_j+\beta_j s+B_j t)\prod_{j=n_1+1}^{p_1}\Gamma^{\mathbf{A}_j}(a_j-\alpha_j s-A_j t)\prod_{j=m_2+1}^{q_2}\Gamma^{\mathbf{D}_j}(1-d_j+\delta_j s-D_j t)}$$

$$\frac{\prod_{j=1}^{n_2}\Gamma^{\mathbf{C}_j}(1-c_j+\gamma_j s-C_j t)}{\prod_{j=1}^{n_2}\Gamma^{\mathbf{C}_j}(c_j-\gamma_j s+C_j t)} \tag{1.2}$$

$$\psi(s)=\frac{\prod_{j=1}^{m_3}\Gamma^{\mathbf{F}_j}(f_j-F_i s)\prod_{j=1}^{n_3}\Gamma^{\mathbf{E}_j}(1-e_j+E_j s)}{\prod_{j=m_3+1}^{q_3}\Gamma^{\mathbf{F}_j}(1-f_j+F_j s)\prod_{j=n_3+1}^{p_3}\Gamma^{\mathbf{E}_j}(e_j-E_j s)} \tag{1.3}$$

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and

$$\phi(t)=\frac{\prod_{j=1}^{m_4}\Gamma^{\mathbf{H}_j}(h_j-H_jt)\prod_{j=1}^{n_4}\Gamma^{\mathbf{G}_j}(1-g_j+G_jt)}{\prod_{j=m_4+1}^{q_4}\Gamma^{\mathbf{H}_j}(1-h_j+H_jt)\prod_{j=n_4+1}^{p_4}\Gamma^{\mathbf{G}_j}(g_j-G_jt)} \tag{1.4}$$

where $z_1$and z_2 are not zero and an empty product is interpreted as unity. Also $\mathbf{m_j},\mathbf{n_j},\mathbf{p_j},\mathbf{q_j}(\mathbf{j}=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4})$ are all positive integers such that $\mathbf{0}\leqslant\mathbf{m_j}\leqslant\mathbf{q_j};\mathbf{0}\leqslant\mathbf{n_j}\leqslant\mathbf{p_j}(\mathbf{j}=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4})$. The letters $\alpha,\beta,\gamma,\delta,A,B,C,D,E,F,G,H$ and $\mathbf{A},\mathbf{B},\mathbf{C}$ and $\mathbf{D}$ are all positive numbers and the letters a, b, c, d, e, f, g, h are complex numbers.
the definition of modified generalized I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the s-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\mathbf{\Gamma^{B_j}(b_j-\beta_js-B_jt)(j=1,\cdots,m_1)}$, $\mathbf{\Gamma^{D_j}(d_i-\delta_js+D_jt)(j=1,\cdots,m_2)}$ and $\Gamma^{\mathbf{F_j}}(f_j-F_js)(j=1,\cdots,m_3)$, are to the right and all the poles of $\Gamma^{\mathbf{A_j}}(1-a_j+\alpha_js+A_jt)(j=1,\cdots,n_1)$, $\Gamma^{\mathbf{E_j}}(1-e_j+E_js)$,$(j=1,\cdots,n_3)$ and $\Gamma^{\mathbf{C_j}}(1-c_j+\gamma_js-C_jt)(j=1,\cdots,n_2)$ lie to the left of L_1. The contour L_2 is in the$t$ -plane and runs from$-\omega\infty$ to$+\omega\infty$ with loops, if necessary, to ensure that the poles of $\mathbf{\Gamma^{B_j}(b_j-\beta_js-B_jt)(i=1,\cdots,m_1)}$, $\Gamma^{\mathbf{H_j}}(h_j-H_jt)(j=1,\cdots,m_4)$ $\Gamma^{\mathbf{C_j}}(1-c_j+\gamma_js-C_jt)(j=1,\cdots,n_2)$ are to the right and all the poles of $\Gamma^{\mathbf{A_j}}(1-a_j+\alpha_js+A_jt)(j=1,\cdots,n_1)$, $\Gamma^{\mathbf{D_j}}(d_j-\delta_js+D_jt)(j=1,\cdots,m_2)$ and $\Gamma^{\mathbf{G_j}}(1-g_j+G_jt)(j=1,\cdots,n_4)$ lie to the left of L_2. The poles of the integrand are assumed to be simple.

The function defined by the equation (3.1) is analytic function of z_1 and z_2 if

$$\sum_{j=1}^{p_1}\mathbf{A}_j\alpha_j+\sum_{j=1}^{p_2}\mathbf{C}_j\gamma_j+\sum_{j=1}^{p_3}\mathbf{E}_jE_j<\sum_{j=1}^{q_1}\mathbf{B}_j+\sum_{j=1}^{q_2}\mathbf{D}_j\delta_j+\sum_{i=1}^{q_3}\mathbf{F}_jF_j \tag{1.5}$$

$$\sum_{j=1}^{p_1}\mathbf{A}_jA_j+\sum_{j=1}^{p_2}\mathbf{D}_jC_j+\sum_{j=1}^{p_4}\mathbf{G}_jG_j<\sum_{j=1}^{q_1}\mathbf{B}_jB_j+\sum_{j=1}^{q_2}\mathbf{C}_jC_j+\sum_{j=1}^{q_4}\mathbf{H}_jH_j \tag{1.6}$$

The integral (3.1) is absolutely convergent if $|argz_1|<\frac{1}{2}U\pi,|argz_2|<\frac{1}{2}V\pi$ where

$$U=\sum_{j=1}^{n_1}\mathbf{A}_j\alpha_j-\sum_{j=n_1+1}^{p_1}\mathbf{A}_j\alpha_j+\sum_{j=1}^{m_1}\beta_j\mathbf{B}_j-\sum_{j=m_1+1}^{q_1}\beta_j\mathbf{B}_j+\sum_{j=1}^{n_2}\mathbf{C}_j\gamma_j-\sum_{j=n_2+1}^{p_2}\mathbf{C}_j\gamma_j+\sum_{j=1}^{m_2}\mathbf{D}_j\delta_j$$

$$-\sum_{j=m_2+1}^{q_2}\mathbf{D}_j\delta_j+\sum_{j=1}^{n_3}\mathbf{E}_jE_j-\sum_{j=n_3+1}^{p_3}\mathbf{E}E_j+\sum_{j=1}^{m_3}\mathbf{F}_jF_j-\sum_{j=m_3+1}^{q_3}\mathbf{F}_jF_j>0 \tag{1.7}$$

$$V-\sum_{j=1}^{n_1}\mathbf{A}_jA_j-\sum_{j=n_1+1}^{p_1}\mathbf{A}_jA_j+\sum_{j=1}^{m_1}B_j\mathbf{B}_j-\sum_{j=m_1+1}^{q_1}B_j\mathbf{B}_j+\sum_{j=1}^{n_2}\mathbf{C}_jC_j-\sum_{j=n_2+1}^{p_2}\mathbf{C}_jC_j+\sum_{j=1}^{m_2}\mathbf{D}_jD_j$$

$$-\sum_{j=m_2+1}^{q_2}\mathbf{D}_j\delta_j+\sum_{j=1}^{n_4}\mathbf{G}_jG_j-\sum_{j=n_4+1}^{p_4}\mathbf{G}_jG_j+\sum_{j=1}^{m_4}\mathbf{H}_jH_j-\sum_{j=m_4+1}^{q_4}\mathbf{H}_jH_j>0 \tag{1.8}$$

We may establish the asymptotic behavior in the following convenient form, see B.L.J. Braaksma [5].

$$I(z_1,z_2)=0(\,|z_1|^{\alpha_1},|z_2|^{\alpha_2}\,)\,,max(\,|z_1|,|z_2|\,)\rightarrow 0$$

$$I(z_1,z_2)=0(\,|z_1|^{\beta_1},|z_2|^{\beta_2}\,)\,,min(\,|z_1|,|z_2|\,)\rightarrow\infty\;:$$

$\alpha_1=\min_{1\leqslant j\leqslant m_3}Re\left[\mathbf{F}_j\left(\frac{f_j}{F_j}\right)\right]$ and $\alpha_2=\min_{1\leqslant j\leqslant m_4}Re\left[\mathbf{H}_j\left(\frac{h_j}{H_j}\right)\right]$

$\beta_1=\max_{1\leqslant j\leqslant n_3}Re\left[\mathbf{E}_j\left(\frac{e_j-1}{E_j}\right)\right]$ and $\beta_2=\max_{1\leqslant j\leqslant n_4}Re\left[\mathbf{G}_j\left(\frac{g_j-1}{G_j}\right)\right]$

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II. REQUIRED INTEGRAL

In this section, we give a generalized finite integral, see Prudnikov et al. ([11], Ch2.2.11, 24 page 316).

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[(z+\sqrt{a-x})\right]^v + [(z-\sqrt{a-x})]^v \mathrm{d}x = 2\sqrt{\pi}\frac{\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}z^{\alpha-\frac{1}{2}}z^v$$

$${}_2F_1\left[\frac{1-v}{2},\frac{-v}{2};\alpha+\frac{1}{2};\frac{a}{z^2}\right] \tag{2.1}$$

where $0 < a, Re(\alpha), z^2 \notin [0,a]$.

III. MAIN INTEGRAL

We have the general result involving an unified finite integral with several parameters involving the modified of generalized I-function of two variables.

Theorem:

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v + \left(z-\sqrt{a-x}\right)^v\right]\mathrm{I}(Zx^b, Z'x^c)\mathrm{d}x = 2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}\left(\frac{1-v}{2}\right)^{n'}\left(\frac{-v}{2}\right)^{n'}$$

$$I^{m_1,n_1+1:m_2,n_2:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_2,q_2:p_3,q_3:p_4,q_4}\left(\begin{array}{c|l} z^bZ & \mathbf{A}_1, \{(a_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1} : \{(c_j;\gamma_j,C_j;\mathbf{C}_j)\}_{1,p_2} : \{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3}, \{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \\ z^cZ' & \{(b_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1}, \mathbf{B}_1 : \{(d_j;\delta_j,D_j;\mathbf{D}_j)\}_{1,q_2} : \{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3}, \{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4} \end{array}\right) \tag{3.1}$$

$$0 < a,b,c, Re(\alpha), z^2 \notin [0,a], 0 < Re(\alpha)+(b+c)\min_{1\leqslant j\leqslant m_3} Re\left(\mathbf{F_j}\frac{f_j}{F_j}\right), 0 < Re(\alpha)+(b+c)\min_{1\leqslant j\leqslant m_4} Re\left(\mathbf{H_j}\frac{h_j}{H_j}\right)$$

$|argz_1| < \frac{1}{2}U\pi$, $|argz_2| < \frac{1}{2}V\pi$, U and V are defined respectively by the equations (1.7) and (1.8).

where

$$\mathbf{A}_1 - (1\text{-}\alpha; b, c; 1);\ \mathbf{B}_1 = \left(\tfrac{1}{2}-\alpha-n'; b, c; 1\right) \tag{3.2}$$

To prove the theorem, expressing the modified generalized I-function of two variables in double Mellin-Barnes contour integral with the help of (1.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process and collecting the power of x , We will write the left hand side of the equation (3.1), L and we have :

$$L = \int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v + \left(z-\sqrt{a-x}\right)^v\right]\ \mathrm{I}(Zx^b, Z'x^c)\mathrm{d}x =$$

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\int_0^a \frac{x^{\alpha+bs+ct-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v + \left(z-\sqrt{a-x}\right)^v\right]\mathrm{d}x\mathrm{d}s\ \mathrm{d}t \tag{3.3}$$

Calculating the x-integral by using the lemma, this gives:

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$$I=\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v+\left(z-\sqrt{a-x}\right)^v\right]\ \mathrm{I}(Zx^b,Z'x^c)\mathrm{d}x=\ 2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}$$

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\frac{\Gamma(\alpha+bs+ct)}{\Gamma(\alpha+bs+ct+\frac{1}{2})}z^{bs+ct}{}_2F_1\left[\frac{1-v}{2},\frac{-v}{2};\alpha+bs+bt+\frac{1}{2};\frac{a}{z^2}\right]\mathrm{d}s\mathrm{d}t \tag{3.4}$$

use the expression of the Gauss hypergeometric function in term of serie $\sum_{n'=0}^{\infty}$, (see Slater [16]), under the hypothesis, we can interchanged this serie and the double (s,t) - integrals, then we apply known the relations $a=\frac{\Gamma(a+1)}{\Gamma(a)}$ and $\Gamma(a)(a)_n=\Gamma(a+n)$ where $a\neq 0,-1,-2,\cdots,$. Then

$$J=\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v+\left(z-\sqrt{a-x}\right)^v\right]\mathrm{I}(Zx^b,Z'x^c)\mathrm{d}x=2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}$$

$$\left(\tfrac{1-v}{2}\right)^{n'}\left(\tfrac{-v}{2}\right)^{n'}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\,z^{bs+ct}\Gamma\left(\alpha+bs+ct\right)\frac{1}{\Gamma\left(\alpha+bs+ct+\frac{1}{2}+n'\right)}\mathrm{d}s\mathrm{d}t \tag{3.5}$$

Interpreting this double integrals contour in term of the modified generalized I-function of two variables, we obtain the desired result. In the following, we will use the notations U_1 and V_1.

IV. SPECIAL CASES

In this section, we study several special cases and remarks. We consider the generalized I-function of two variables, in this situation, we have the conditions: $m_2=n_2=p_2=q_2=0$ and

Corollary 1

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v+\left(z-\sqrt{a-x}\right)^v\right]\mathrm{I}(Zx^b,Z'x^c)\mathrm{d}x=2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}\left(\tfrac{1-v}{2}\right)^{n'}\left(\tfrac{-v}{2}\right)^{n'}$$

$$I^{m_1,n_1+1:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} z^bZ & \mathbf{A}_1,\{(a_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1}:\{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3},\{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \cdot & \\ z^cZ' & \{(b_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1},\mathbf{B}_1:\{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3},\{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4}\end{array}\right) \tag{4.1}$$

The conditions and notations mentionned in the theorem are satisfied with the conditions : $m_2=n_2=p_2=q_2=0$ and

$|argz_1|<\frac{1}{2}U_1\pi,|argz_2|<\frac{1}{2}V_1\pi,$ U_1 where

$$U_1=\sum_{j=1}^{n_1}\mathbf{A}_j\alpha_j-\sum_{j=n_1+1}^{p_1}\mathbf{A}_j\alpha_j+\sum_{j=1}^{m_1}\beta_j\mathbf{B}_j-\sum_{j=m_1+1}^{q_1}\beta_j\mathbf{B}_j+\sum_{j=1}^{n_3}\mathbf{E}_jE_j-\sum_{j=n_3+1}^{p_3}\mathbf{E}_jE_j+\sum_{j=1}^{m_3}\mathbf{F}_jF_j-\sum_{j=m_3+1}^{q_3}\mathbf{F}_jF_j>0 \tag{4.2}$$

$$V_1=\sum_{j=1}^{n_1}\mathbf{A}_jA_j-\sum_{j=n_1+1}^{p_1}\mathbf{A}_jA_j+\sum_{j=1}^{m_1}B_j\mathbf{B}_j-\sum_{j=m_1+1}^{q_1}B_j\mathbf{B}_j+\sum_{j=1}^{n_4}\mathbf{G}_jG_j-\sum_{j=n_4+1}^{p_4}\mathbf{G}_jG_j+\sum_{j=1}^{m_4}\mathbf{H}_jH_j-\sum_{j=m_4+1}^{q_4}\mathbf{H}_jH_j>0 \tag{4.3}$$

Now, we consider the above corollary with m_1 0. the function defined in the beginning reduces to I-function of two variables defined by Kumari et al. [8], this gives.

Corollary 2

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v+\left(z-\sqrt{a-x}\right)^v\right]\mathrm{I}(Zx^b, Z'x^c)\mathrm{d}x = 2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}\left(\frac{1-v}{2}\right)^{n'}\left(\frac{-v}{2}\right)^{n'}$$

$$I^{0,n_1+1:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} z^bZ & \mathbf{A}_1, \{(a_j;\alpha_j, A_j;\mathbf{A}_j)\}_{1,p_1} : \{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3}, \{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ z^cZ' & \{(\mathrm{b}_j;\beta_j, B_j;\mathbf{B}_j)\}_{1,q_1}, \mathbf{B}_1 : \{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3}, \{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4} \end{array}\right) \quad (4.4)$$

By considering the conditions of the corollary 1 with $\mathrm{m}_1 = 0$, and $|argz_1| < \frac{1}{2}U_1'\pi, |argz_2| < \frac{1}{2}V_1'\pi$, U_1' where V_1',

$$U_1' = \sum_{j=1}^{n_1}\mathbf{A}_j\alpha_j - \sum_{j=n_1+1}^{p_1}\mathbf{A}_j\alpha_j - \sum_{j=1}^{q_1}\beta_j\mathbf{B}_j + \sum_{j=1}^{n_3}\mathbf{E}_jE_j \qquad \sum_{j=m_3+1}^{q_3}\mathbf{F}_jF_j > 0 \quad (4.5)$$

$$\sum_{j=n_3+1}^{p_3}\mathbf{E}_jE_j + \sum_{j=1}^{m_3}\mathbf{F}_jF_j \quad (4.6)$$

$$V_1' = \sum_{j=1}^{n_1}\mathbf{A}_jA_j \quad \sum_{j=n_1+1}^{p_1}\mathbf{A}_jA_j - \sum_{j=1}^{q_1}B_j\mathbf{B}_j + \sum_{j=1}^{n_4}\mathbf{G}_jG_j - \sum_{j=n_4+1}^{p_4}\mathbf{G}_jG_j + \sum_{j=1}^{m_4}\mathbf{H}_jH_j - \sum_{j=m_4+1}^{q_4}\mathbf{H}_jH_j > 0$$

The modified generalized I-function reduces to Modified generalized H-function defined by Prasad and Prasad [10]. In this cases, we have : $\mathbf{A_j} = \mathbf{B_j} = \mathbf{C_j} = \mathbf{D_j} = \mathbf{E_j} = \mathbf{F_j} = \mathbf{G_j} = \mathbf{H_j} = 1$ and

Corollary 3

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v+\left(z-\sqrt{a-x}\right)^v\right]\mathrm{H}\,(Zx^b, Z'x^c)\mathrm{d}x = 2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}\left(\frac{1-v}{2}\right)^{n'}\left(\frac{-v}{2}\right)^{n'}$$

$$H^{m_1,n_1+1:m_2,n_2:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_2,q_2:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} z^bZ & \mathbf{A}_1', \{(a_j;\alpha_j, A_j)\}_{1,p_1} : \{(c_j;\gamma_j, C_j)\}_{1,p_2} : \{(e_j;E_j)\}_{1,p_3}, \{(g_j;G_j)\}_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ z^cZ' & \{(\mathrm{b}_j;\beta_j, B_j)\}_{1,q_1}, \mathbf{B}_1' : \{(d_j;\delta_j, D_j)\}_{1,q_2} : \{(f_j;F_j)\}_{1,q_3}, \{(h_j;H_j)\}_{1,q_4} \end{array}\right) \quad (4.7)$$

By respecting the conditions and notations of the theorem with $\mathbf{A_j} = \mathbf{B_j} = \mathbf{C_j} = \mathbf{D_j} = \mathbf{E_j} = \mathbf{F_j} = \mathbf{G_j} = \mathbf{H_j} = 1$ and

$|argz_1| < \frac{1}{2}U_1\pi, |argz_2| < \frac{1}{2}V_1\pi$, U_1 and V_1 are defined by the following formulas:

$$U_1 = \sum_{j=1}^{n_1}\alpha_j - \sum_{j=n_1+1}^{p_1}\alpha_j + \sum_{j=1}^{m_1}\beta_j - \sum_{j=m_1+1}^{q_1}\beta_j + \sum_{j=1}^{n_2}\gamma_j - \sum_{j=n_2+1}^{p_2}\gamma_j + \sum_{j=1}^{m_2}\delta_j$$

$$-\sum_{j=m_2+1}^{q_2}\delta_j + \sum_{j=1}^{n_3}E_j - \sum_{j=n_3+1}^{p_3}E_j + \sum_{j=1}^{m_3}F_j - \sum_{j=m_3+1}^{q_3}F_j > 0 \quad (4.8)$$

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$$V_1 - \sum_{j=1}^{n_1} A_j \quad \sum_{j=n_1+1}^{p_1} A_j + \sum_{j=1}^{m_1} B_j \quad \sum_{j=m_1+1}^{q_1} B_j + \sum_{j=1}^{n_2} C_j \quad \sum_{j=n_2+1}^{p_2} C_j + \sum_{j=1}^{m_2} D_j$$

$$- \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_4} G_j - \sum_{j=n_4+1}^{p_4} G_j + \sum_{j=1}^{m_4} H_j - \sum_{j=m_4+1}^{q_4} H_j > 0 \tag{4.9}$$

where

$$\mathbf{A}_1' = (1\text{-}\alpha; b, c); \;\; \mathbf{B}_1' = \left(\tfrac{1}{2} - \alpha - n'; b, c\right) \tag{4.10}$$

The generalized modified H-function reduces to the generalized modified of G-function of two variables defined Agarwal [1], we suppose : $\mathrm{b} - \mathrm{c} - 1$, then $\mathrm{A}_1' = (1-\alpha)$, $\left(\frac{1}{2} - \alpha\right)$, $\mathrm{B}_1' = \left(\frac{1}{2} - \alpha - n'\right)$ and we have the conditions :

$$(\alpha_j)_{1,p_1} = (A_j)_{1,p_1} = (\gamma_j)_{1,p_2} = (C_j)_{1,p_2} = (E_j)_{1,p_3} = (G_j)_{1,p_4} = 1$$

$$= (\beta_j)_{1,q_1} = (B_j)_{1,q_1} = (\delta_j)_{1,q_2} = (D_j)_{1,q_2} = (F_j)_{1,q_3} = (H_j)_{1,q_4}$$

and the result:

Corollary 4

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}} \left[\left(z+\sqrt{a-x}\right)^v + \left(z-\sqrt{a-x}\right)^v\right] \mathrm{G}\,(Zx, Z'x)\mathrm{d}x = 2\sqrt{\pi} z^{\alpha-\frac{1}{2}+v} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!} \left(\tfrac{1-v}{2}\right)^{n'} \left(\tfrac{-v}{2}\right)^{n'}$$

$$G^{m_1,n_1+1:m_2,n_2:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_2,q_2:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} \mathrm{Z} & \mathrm{A}_1', (a_1)_{1,p_1} : (c_j)_{1,p_2} : (e_j)_{1,p_3}, (g_j)_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}' & (\mathrm{b}_j)_{1,q_1}, B_1' : (d_j)_{1,q_2} : (f_j)_{1,q_3}, (h_j)_{1,q_4} \end{array}\right) \tag{4.11}$$

under the conditions verified by the corollary 3, about the modified generalized H-function of two variables and $(\alpha_j)_{1,p_1} = (A_j)_{1,p_1} = (\gamma_j)_{1,p_2} = (C_j)_{1,p_2} = (E_j)_{1,p_3} = (G_j)_{1,p_4} = 1$ and $|argz_1| < \frac{1}{2}U_1\pi, |argz_2| < \frac{1}{2}V_1\pi$, U_1 and V_1 are defined by the following formulas :

$$\mathrm{U}_1 = \left[m_1 + n_1 + m_2 + n_2 + m_3 + n_3 - \tfrac{1}{2}(p_1 + q_1 + p_2 + q_2 + p_3 + q_3)\right] \tag{4.12}$$

and

$$\mathrm{V}_1 - \left[m_1 + n_1 + m_2 + n_2 + m_4 + n_4 - \tfrac{1}{2}(p_1 + q_1 + p_2 + q_2 + p_4 + q_4)\right] \tag{4.13}$$

In the following, we suppose $\mathrm{m}_1 = m_2 = n_2 = p_2 = q_2 = 0$, inthis situation, we get the H-function of two variables defined by K.C. Gupta, and P.K. Mittal [6] : We consider the corollary 2 (I-function of two variables studied by Kumari et al. [5] and let : $\mathbf{A_j} = \mathbf{C_j} = \mathbf{D_j} = \mathbf{F_j} = \mathbf{G_j} = \mathbf{H_j} = 1$, we obtain the relation :

Corollary 5

$$\int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}} \left[\left(z+\sqrt{a-x}\right)^v + \left(z-\sqrt{a-x}\right)^v\right] \mathrm{H}\,(Zx^b, Z'x^c)\mathrm{d}x = 2\sqrt{\pi} z^{\alpha-\frac{1}{2}+v} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!} \left(\tfrac{1-v}{2}\right)^{n'} \left(\tfrac{-v}{2}\right)^{n'}$$

$$H^{0,n_1+1:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} z^aZ & \mathbf{A}'_1,\{(a_j;\alpha_j,A_j)\}_{1,p_1}:\{(e_j;E_j)\}_{1,p_3},\{(g_j;G_j)\}_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ z^cZ' & \{(b_j;\beta_j,B_j)\}_{1,q_1},\mathbf{B}'_1:\{(f_j;F_j)\}_{1,q_3},\{(h_j;H_j)\}_{1,q_4} \end{array}\right) \quad (4.14)$$

The conditions and notations are satisfied by the corollary 2 with the conditions $\mathbf{A_j}=\mathbf{C_j}=\mathbf{D_j}=\mathbf{F_j}=\mathbf{G_j}=\mathbf{H_j}=1$ and

$$arg z_1 < \frac{1}{2}U''_1\pi, |arg z_2| < \frac{1}{2}V''_1\pi,$$ U''_1 where V''_1.

$$U''_1=\sum_{j=1}^{n_1}\alpha_j-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j>0 \quad (4.15)$$

$$V''_1=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{n_4}G_j-\sum_{j=n_4+1}^{p_4}G_j+\sum_{j=1}^{m_4}H_j-\sum_{j=m_4+1}^{q_4}H_j>0 \quad (4.16)$$

The quantities $\mathbf{A}'_1$ and $\mathbf{B}'_1$ are defined by the equation (4.10).
Taking conditions :
$m_1=0,\ (\alpha_j)_{1,p_1}=(A_j)_{1,p_1}=(E_j)_{1,p_3}=(G_j)_{1,p_4}=1 \ =(\beta_j)_{1,q_1}=(B_j)_{1,q_1}=(F_j)_{1,q_3}=(H_j)_{1,q_4}$, we have the formula with the G-function of two variables defined by Agarwal [1].

Corollary 6

$$\int_0^a\frac{x^{\alpha-1}}{\sqrt{a-x}}\left[\left(z+\sqrt{a-x}\right)^v+\left(z-\sqrt{a-x}\right)^v\right]\ \mathrm{G}\,(Zx,Z'x)\mathrm{d}x = 2\sqrt{\pi}z^{\alpha-\frac{1}{2}+v}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}\left(\frac{1-v}{2}\right)^{n'}\left(\frac{-v}{2}\right)^{n'}$$

$$G^{0,n_1+1:m_3,n_3:m_4,n_4}_{p_1+1,q_1+1:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} \mathrm{Z} & \mathrm{A}'_1,(a_1)_{1,p_1}:(e_j)_{1,p_3},(g_j)_{1,p_4} \\ \mathrm{Z}' & (\mathrm{b}_j)_{1,q_1},B'_1:(f_j)_{1,q_3},(h_j)_{1,q_4} \end{array}\right) \quad (4.17)$$

By using the same notations and conditions that the above corollary and the following conditions are respected.
$m_1=0$: $(\alpha_j)_{1,p_1}=(A_j)_{1,p_1}=(E_j)_{1,p_3}=(G_j)_{1,p_4}=1\ =(\beta_j)_{1,q_1}=(B_j)_{1,q_1}=(F_j)_{1,q_3}=(H_j)_{1,q_4}$ and

$|arg z_1| < \frac{1}{2}U_1\pi, |arg z_2| < \frac{1}{2}V_1\pi$, U_1 and V_1 are defined by the following formulas :

$$\mathrm{U}_1=\left[n_1+m_3\ \ +n_3-\tfrac{1}{2}(p_1+q_1+p_3+q_3)\right] \quad (4.18)$$

and

$$\mathrm{V}_1\ \ \left[n_1+m_4+n_4-\tfrac{1}{2}(p_1+q_1+p_4+q_4)\right] \quad (4.19)$$

Remarks

We have the same generalized finite integral with the modified generalized of the Aleph-function of two variables defined by Kumar [7] and Sharma [14] and the special cases, the I-function defined by Saxena [13] by Rathie [12], the Fox's H-function, We have the same generalized multiple finite integrals with the incomplete aleph-function defined byBansal et al. [3], the incomplete I-function studied by Bansal and Kumar. [2] and the incomplete Fox's H-function givenby Bansal et al. [4], the Psi function defined by Pragathi et al. [9].

V. CONCLUSION

The importance of our all the results lies in their manifold generality. First by specializing the various parameters as well as variable of the generalized modified I-function of two variables, we obtain a big

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number of results involving remarkably wide variety of useful special functions (or product of such special functions) which are expressible in terms of I-function of two variables or one variable defined by Rathie [12], H-function of two or one variables, Meijer's G-function of two or one variables and hypergeometric function of two or one variables. Secondly, by specializing the parameters of this unified finite integral, we can get a large number of integrals involving the modified generalized I-functions of two variables and the others functions seen in this document.

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