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Infinite Integral Involving the Generalized Modified I-Function of Two Variables

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ABSTRACT

In the present paper, we evaluate the general infinite integral involving the generalized modified I-functions of two variables. At the end, we shall see several corollaries and remarks.

Keywords- Incomplete Gamma function, modified generalized I-function of two variables, generalized I-function of two variables, generalized modified H-function of two variables, generalized modified Meijer-function of two variables, I-function of two variables, H-function of two variables, Meijer-function of two variables, double Mellin-Barnes integrals contour, finite integral.

2010 Mathematics Subject Classification: 33C05, 33C60.

I. INTRODUCTION AND PRELIMINARIES

Kumari et al. [8] have defined the I-function of two variables. On the other hand Prasad and Prasad [10] have studied the modified generalized H-function of two variables. Recently Singh and Kumar [15] have worked about the modified of generalized I-function of two variables. This function is an extension of the I-function of two variables and modified of generalized H-function of two variables. In this paper, first, we define the modified generalized I-function of two variables. Then we calculate the finite integral involving this function. At the last section, we will see several cases and remarks. The double Mellin-Barnes integrals contour occurring in this paper will be referred to as the generalized modified I-function of two variables throughout our present study and will be defined and represented as follows:

We have

$$I(z_1,z_2)=I^{m_1,n_1,m_2,n_2,m_3,n_3,m_4,n_4}_{p_1,q_1,p_2,q_2;p_3,q_3;p_4,q_4}\left(\begin{array}{c|c} z_1 & \{(a_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1}:\{(c_j;\gamma_j,C_j;\mathbf{C}_j)\}_{1,p_2}:\{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3},\{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_2 & \{(b_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1}:\{(d_j;\delta_j,D_j;\mathbf{D}_j)\}_{1,q_2}:\{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3},\{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)z_1^s z_2^t\,\mathrm{d}s\mathrm{d}t \tag{1.1}$$

where

$$\theta(s,t)=\frac{\prod_{j=1}^{m_1}\Gamma^{\mathbf{B}_j}(b_j-\beta_j s-B_j t)\prod_{j=1}^{n_1}\Gamma^{\mathbf{A}_j}(1-a_j+\alpha_j s+A_j t)\prod_{j=1}^{m_2}\Gamma^{\mathbf{D}_j}(d_j-\delta_j s+D_j t)}{\prod_{j=m_1+1}^{q_1}\Gamma^{\mathbf{B}_j}(1-b_j+\beta_j s+B_j t)\prod_{j=n_1+1}^{p_1}\Gamma^{\mathbf{A}_j}(a_j-\alpha_j s-A_j t)\prod_{j=m_2+1}^{q_2}\Gamma^{\mathbf{D}_j}(1-d_j+\delta_j s-D_j t)}$$

$$\frac{\prod_{j=1}^{n_2}\Gamma^{\mathbf{C}_j}(1-c_j+\gamma_j s-C_j t)}{\prod_{j=1}^{n_2}\Gamma^{\mathbf{C}_j}(c_j-\gamma_j s+C_j t)} \tag{1.2}$$

$$\psi(s)=\frac{\prod_{j=1}^{m_3}\Gamma^{\mathbf{F}_j}(f_j-F_i s)\prod_{j=1}^{n_3}\Gamma^{\mathbf{E}_j}(1-e_j+E_j s)}{\prod_{j=m_3+1}^{q_3}\Gamma^{\mathbf{F}_j}(1-f_j+F_j s)\prod_{j=n_3+1}^{p_3}\Gamma^{\mathbf{E}_j}(e_j-E_j s)} \tag{1.3}$$

and

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$$\phi(t)=\frac{\prod_{j=1}^{m_4}\Gamma^{\mathbf{H}_j}(h_j-H_jt)\prod_{j=1}^{n_4}\Gamma^{\mathbf{G}_j}(1-g_j+G_jt)}{\prod_{j=m_4+1}^{q_4}\Gamma^{\mathbf{H}_j}(1-h_j+H_jt)\prod_{j=n_4+1}^{p_4}\Gamma^{\mathbf{G}_j}(g_j-G_jt)} \tag{1.4}$$

where $z_1$and z_2 are not zero and an empty product is interpreted as unity. Also $\mathbf{m_j},\mathbf{n_j},\mathbf{p_j},\mathbf{q_j}(\mathbf{j}=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4})$ are all positive integers such that $\mathbf{0}\leqslant\mathbf{m_j}\leqslant\mathbf{q_j};\mathbf{0}\leqslant\mathbf{n_j}\leqslant\mathbf{p_j}(\mathbf{j}=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4})$. The letters $\alpha,\beta,\gamma,\delta,A,B,C,D,E,F,G,H$ and $\mathbf{A},\mathbf{B},\mathbf{C}$ and $\mathbf{D}$ are all positive numbers and the letters a, b, c, d, e, f, g, h are complex numbers.
the definition of modified generalized I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the s-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\mathbf{\Gamma^{B_j}(b_j-\beta_js-B_jt)(j=1,\cdots,m_1)}$, $\mathbf{\Gamma^{D_j}(d_i-\delta_js+D_jt)(j=1,\cdots,m_2)}$ and $\Gamma^{\mathbf{F_j}}(f_j-F_js)(j=1,\cdots,m_3)$, are to the right and all the poles of $\Gamma^{\mathbf{A_j}}(1-a_j+\alpha_js+A_jt)(j=1,\cdots,n_1)$, $\Gamma^{\mathbf{E_j}}(1-e_j+E_js),(j=1,\cdots,n_3)$ and $\Gamma^{\mathbf{C_j}}(1-c_j+\gamma_js-C_jt)(j=1,\cdots,n_2)$ lie to the left of L_1. The contour L_2 is in the t-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\mathbf{\Gamma^{B_j}(b_j-\beta_js-B_jt)(i=1,\cdots,m_1)}$, $\Gamma^{\mathbf{H_j}}(h_j-H_jt)(j=1,\cdots,m_4)\,\Gamma^{\mathbf{C_j}}(1-c_j+\gamma_js-C_jt)(j=1,\cdots,n_2)$ are to the right and all the poles of $\Gamma^{\mathbf{A_j}}(1-a_j+\alpha_js+A_jt)(j=1,\cdots,n_1)$, $\Gamma^{\mathbf{D_j}}(d_j-\delta_js+D_jt)(j=1,\cdots,m_2)$ and $\Gamma^{\mathbf{G_j}}(1-g_j+G_jt)(j=1,\cdots,n_4)$ lie to the left of L_2. The poles of the integrand are assumed to be simple.

The function defined by the equation (3.1) is analytic function of z_1 and z_2 if

$$\sum_{j=1}^{p_1}\mathbf{A}_j\alpha_j+\sum_{j=1}^{p_2}\mathbf{C}_j\gamma_j+\sum_{j=1}^{p_3}\mathbf{E}_jE_j<\sum_{j=1}^{q_1}\mathbf{B}_j+\sum_{j=1}^{q_2}\mathbf{D}_j\delta_j+\sum_{i=1}^{q_3}\mathbf{F}_jF_j \tag{1.5}$$

$$\sum_{j=1}^{p_1}\mathbf{A}_jA_j+\sum_{j=1}^{p_2}\mathbf{D}_jC_j+\sum_{j=1}^{p_4}\mathbf{G}_jG_j<\sum_{j=1}^{q_1}\mathbf{B}_jB_j+\sum_{j=1}^{q_2}\mathbf{C}_jC_j+\sum_{j=1}^{q_4}\mathbf{H}_jH_j \tag{1.6}$$

The integral (3.1) is absolutly convergent if $|argz_1|<\frac{1}{2}U\pi,|argz_2|<\frac{1}{2}V\pi$ where

$$U=\sum_{j=1}^{n_1}\mathbf{A}_j\alpha_j-\sum_{j=n_1+1}^{p_1}\mathbf{A}_j\alpha_j+\sum_{j=1}^{m_1}\beta_j\mathbf{B}_j-\sum_{j=m_1+1}^{q_1}\beta_j\mathbf{B}_j+\sum_{j=1}^{n_2}\mathbf{C}_j\gamma_j-\sum_{j=n_2+1}^{p_2}\mathbf{C}_j\gamma_j+\sum_{j=1}^{m_2}\mathbf{D}_j\delta_j$$

$$-\sum_{j=m_2+1}^{q_2}\mathbf{D}_j\delta_j+\sum_{j=1}^{n_3}\mathbf{E}_jE_j-\sum_{j=n_3+1}^{p_3}\mathbf{E}E_j+\sum_{j=1}^{m_3}\mathbf{F}_jF_j-\sum_{j=m_3+1}^{q_3}\mathbf{F}_jF_j>0 \tag{1.7}$$

$$V=\sum_{j=1}^{n_1}\mathbf{A}_jA_j-\sum_{j=n_1+1}^{p_1}\mathbf{A}_jA_j+\sum_{j=1}^{m_1}B_j\mathbf{B}_j-\sum_{j=m_1+1}^{q_1}B_j\mathbf{B}_j+\sum_{j=1}^{n_2}\mathbf{C}_jC_j-\sum_{j=n_2+1}^{p_2}\mathbf{C}_jC_j+\sum_{j=1}^{m_2}\mathbf{D}_jD_j$$

$$-\sum_{j=m_2+1}^{q_2}\mathbf{D}_j\delta_j+\sum_{j=1}^{n_4}\mathbf{G}_jG_j-\sum_{j=n_4+1}^{p_4}\mathbf{G}_jG_j+\sum_{j=1}^{m_4}\mathbf{H}_jH_j-\sum_{j=m_4+1}^{q_4}\mathbf{H}_jH_j>0 \tag{1.8}$$

We may establish the the asymptotic behavior in the following convenient form, see B.L.J. Braaksma [5].

$I(z_1,z_2)=0(\,|z_1|^{\alpha_1},|z_2|^{\alpha_2}\,)\,,max(\,|z_1|,|z_2|\,)\rightarrow 0$

$I(z_1,z_2)=0(\,|z_1|^{\beta_1},|z_2|^{\beta_2}\,)\,,min(\,|z_1|,|z_2\,|\,)\rightarrow\infty$:

$\alpha_1=\min\limits_{1\leqslant j\leqslant m_3}Re\left[\mathbf{F}_j\left(\frac{f_j}{F_j}\right)\right]$ and $\alpha_2=\min\limits_{1\leqslant j\leqslant m_4}Re\left[\mathbf{H}_j\left(\frac{h_j}{H_j}\right)\right]$

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$\beta_1 = \max_{1\leqslant j\leqslant n_2} Re\left[\mathbf{E}_j\left(\frac{e_j-1}{E_j}\right)\right]$ and $\beta_2 = \max_{1\leqslant j\leqslant n_4} Re\left[\mathbf{G}_j\left(\frac{g_j-1}{G_j}\right)\right]$

II. REQUIRED INTEGRAL

In this section, we give an generalized infinite integral, see Prudnikov et al. ([11], Ch2.2.11, 27 page 316).

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^\upsilon}\mathrm{d}x = 2^{-\upsilon-1}z^{\alpha-\upsilon-2}B\left(\frac{\upsilon-\alpha}{2}+1,\frac{\alpha}{2}\right)$$

$${}_2F_1\left[\frac{\alpha}{2},\frac{\upsilon+1}{2};1+\upsilon;1-y^2\right] \tag{2.1}$$

where

$0 < Re(y), Re(z); 0 < Re(\alpha) < Re(\upsilon)+2.$

III. MAIN INTEGRAL

Let,

$$X_{\alpha,\upsilon} = \frac{x^\alpha}{(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^\upsilon} \tag{3.1}$$

We have the general result with an unified infinite integrals with several parameters involving the modified of generalized I-function of two variables.

Theorem 1

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^\upsilon} I(ZX_{a,b},Z'X_{c,d})\mathrm{d}x =$$

$$2^{-\upsilon-1}z^{\alpha-\upsilon-2}\sum_{n'=0}^\infty \frac{(1-y^2)^{n'}}{n'!} I^{m_1,n_1+4:m_2,n_2:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_2,q_2:p_3,q_3:p_4,q_4}$$

$$\left(\begin{array}{c|c} 2^{-b}z^{a-b}Z & \mathbf{A}_1,\{(a_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1}:\{(c_j;\gamma_j,C_j;\mathbf{C}_j)\}_{1,p_2}:\{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3},\{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{-d}z^{c-d}Z' & \{(\mathrm{b}_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1},\mathbf{B}_1:\{(d_j;\delta_j,D_j;\mathbf{D}_j)\}_{1,q_2}:\{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3},\{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4} \end{array}\right) \tag{3.2}$$

$0 < a, x', b, c, d, Re(\alpha), z^2 \notin [0,a]$, $0 < Re(\alpha)-(b-a)\min_{1\leqslant j\leqslant m_3} Re\left(\mathbf{F_j}\frac{f_j}{F_j}\right)$, $0 < Re(\alpha)-(d-c)\min_{1\leqslant j\leqslant m_4} Re\left(\mathbf{H_j}\frac{h_j}{H_j}\right)$ $b-a, d-c > 0$ where

$|argz_1| < \frac{1}{2}U\pi, |argz_2| < \frac{1}{2}V\pi$, U and V are defined respectively by the equations (1.7) and (1.8).

$$\mathbf{A}_1 = \left(\tfrac{\alpha-\upsilon}{2};\tfrac{b-a}{2},\tfrac{d-c}{2};1\right), \left(1-\tfrac{\alpha}{2}-n';\tfrac{a}{2},\tfrac{c}{2};1\right), \left(1-\tfrac{\upsilon+1}{2}-n';\tfrac{b}{2},\tfrac{d}{2};1\right), (-\upsilon;b,d;1) \tag{3.3}$$

$$\mathbf{B}_1 = \left(-\tfrac{\upsilon}{2};\tfrac{b}{2},\tfrac{d}{2};1\right),\ \left(1-\tfrac{\upsilon+1}{2};\tfrac{b}{2},\tfrac{d}{2};1\right), (-\upsilon-n';b,d;1) \tag{3.4}$$

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To prove the theorem, expressing the modified generalized I-function of two variables by the double Mellin-Barnes contour integral with the help of (1.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Expressing the quantity $X_{(,)}$ in function of x and collecting the power of x, this gives I

$$I=\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}} I(ZX_{a,b},Z'X_{c,d})\mathrm{d}x=$$

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\int_0^\infty \frac{x^{\alpha+as+ct-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon+bs+dt}}\ \mathrm{dx\ ds\ dt}$$

By using the lemma, calculating the x-integral by using the lemma, this gives:

$$I=\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}} I(ZX_{a,b},Z'X_{c,d})\mathrm{d}x=$$

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\,2^{-bs-dt}z^{as+ct-bs-dt}$$

$$B\left(\frac{\upsilon+bs+dt-\alpha-as-ct}{2}+1,\frac{\alpha+as+ct}{2}\right){}_2F_1\left[\frac{\alpha+as+ct}{2},\frac{\upsilon+bs+dt+1}{2};1+\upsilon+bs+dt;1-y^2\right]\mathrm{ds\ dt}\quad(3.6)$$

Using the definition of the Beta-function, we get

$$I=\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}} I(ZX_{a,b},Z'X_{c,d})\mathrm{d}x=$$

$$\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\,2^{bs+dt}z^{-as-ct+bs+dt}\ \frac{\Gamma\left(\frac{\upsilon-\alpha-as-ct+bs+dt}{2}+1\right)\Gamma\left(\frac{\alpha+as+ct}{2}\right)}{\left(\frac{\upsilon+bs+dt}{2}+1\right)}$$

$${}_2F_1\left[\frac{\alpha+as+ct}{2},\frac{\upsilon+bs+dt+1}{2};1+\upsilon+bs+dt;1-y^2\right]\mathrm{dt}\qquad(3.7)$$

use the expression of the Gauss hypergeometric function in term of serie $\sum_{n'=0}^{\infty}$, (see Slater [16]), under the hypothesis, we can interchanged this serie and the (s,t)- integrals, then we apply the relation $a=\frac{\Gamma(a+1)}{\Gamma(a)}$ and $\Gamma(a)(a)_n=\Gamma(a+n)$where $Re(a)>0$, then

$$I=\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}} I(ZX_{a,b},Z'X_{c,d})\mathrm{d}x=\ 2^{-\upsilon-1}z^{\alpha-\upsilon-2}$$

$$\sum_{n'=0}^{\infty}\frac{(1-y^2)^{n'}}{n'!}\ \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s,t)\psi(s)\phi(t)Z^sZ'^t\,2^{-bs-dt}z^{as+ct-bs-dt}$$

$$\frac{\Gamma\left(\frac{\upsilon-\alpha-as-ct+bs+dt}{2}+1\right)}{\Gamma\left(\frac{\upsilon+bs+dt}{2}+1\right)}\Gamma\left(\frac{\alpha+as+ct}{2}+n'\right)\frac{\Gamma\left(\frac{\upsilon+bs+dt+1}{2}+n'\right)}{\Gamma\left(\frac{\upsilon+1+bs+dt}{2}\right)}\frac{\Gamma(1+\upsilon+bs+dt)}{\Gamma(1+\upsilon+n'+bs+dt)}\mathrm{dt}\qquad(3.8)$$

Now, we given several particular cases and remarks.

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IV. SPECIAL CASES

We consider the generalized I-function of two variables, in this situation, we have $m_2 = n_2 = p_2 = q_2 = 0$ and

Corollary 1

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}} I(ZX_{a,b}, Z'X_{c,d})\mathrm{d}x =$$

$$2\sqrt{\pi} z^{\alpha-\frac{1}{2}+\upsilon} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!} \left(\frac{1-\upsilon}{2}\right)^{n'} \left(\frac{-\upsilon}{2}\right)^{n'}$$

$$I^{m_1,n_1+4:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} 2^{-b}z^{a-b}Z & \mathbf{A}_1, \{(a_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1} : \{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3}, \{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \cdot & \cdot \\ 2^{-c}z^{c-d}Z' & \{(\mathrm{b}_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1}, \mathbf{B}_1 : \{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3}, \{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4} \end{array}\right) \quad (4.1)$$

Provided that : $\mathrm{m}_2 = n_2 = p_2 = q_2 = 0, b-a, d-c > 0$ where

$$U_1 = \sum_{j=1}^{n_1} \mathbf{A}_j\alpha_j - \sum_{j=n_1+1}^{p_1} \mathbf{A}_j\alpha_j + \sum_{j=1}^{m_1} \beta_j\mathbf{B}_j - \sum_{j=m_1+1}^{q_1} \beta_j\mathbf{B}_j + \sum_{j=1}^{n_3} \mathbf{E}_jE_j - \sum_{j=n_3+1}^{p_3} \mathbf{E}E_j + \sum_{j=1}^{m_3} \mathbf{F}_jF_j - \sum_{j=m_3+1}^{q_3} \mathbf{F}_jF_j > 0 \quad (4.2)$$

$$V_1 = \sum_{j=1}^{n_1} \mathbf{A}_jA_j - \sum_{j=n_1+1}^{p_1} \mathbf{A}_jA_j + \sum_{j=1}^{m_1} B_j\mathbf{B}_j - \sum_{j=m_1+1}^{q_1} B_j\mathbf{B}_j + \sum_{j=1}^{n_4} \mathbf{G}_jG_j - \sum_{j=n_4+1}^{p_4} \mathbf{G}_jG_j + \sum_{j=1}^{m_4} \mathbf{H}_jH_j - \sum_{j=m_4+1}^{q_4} \mathbf{H}_jH_j > 0 \quad (4.3)$$

Now, we consider the above corollary with m_1=0 the function defined in the beginning reduces to I-function of two variables defined by Kumari et al. [8], this gives.

Corollary 2

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}} I(ZX_{a,b}, Z'X_{c,d})\mathrm{d}x = 2\sqrt{\pi} z^{\alpha-\frac{1}{2}+\upsilon} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!}$$

$$\left(\frac{1-\upsilon}{2}\right)^{n'} \left(\frac{-\upsilon}{2}\right)^{n'} I^{0,n_1+4:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_3,q_3:p_4,q_4}$$

$$\left(\begin{array}{c|c} 2^{-b}z^{a-b}Z & \mathbf{A}_1, \{(a_j;\alpha_j,A_j;\mathbf{A}_j)\}_{1,p_1} : \{(e_j;E_j;\mathbf{E}_j)\}_{1,p_3}, \{(g_j;G_j;\mathbf{G}_j)\}_{1,p_4} \\ \cdot & \cdot \\ 2^{-d}z^{c-d}Z' & \{(\mathrm{b}_j;\beta_j,B_j;\mathbf{B}_j)\}_{1,q_1}, \mathbf{B}_1 : \{(f_j;F_j;\mathbf{F}_j)\}_{1,q_3}, \{(h_j;H_j;\mathbf{H}_j)\}_{1,q_4} \end{array}\right) \quad (4.4)$$

Under the conditions verified by the first corollary and $\mathrm{m}_1 = 0$, and $|argz_1| < \frac{1}{2}U_1'\pi, |argz_2| < \frac{1}{2}V_1'\pi$, U_1' V_1' are defined as follows.

$$U_1' = \sum_{j=1}^{n_1} \mathbf{A}_j\alpha_j - \sum_{j=n_1+1}^{p_1} \mathbf{A}_j\alpha_j - \sum_{j=1}^{q_1} \beta_j\mathbf{B}_j + \sum_{j=1}^{n_3} \mathbf{E}_jE_j - \sum_{j=n_3+1}^{p_3} \mathbf{E}E_j + \sum_{j=1}^{m_3} \mathbf{F}_jF_j - \sum_{j=m_3+1}^{q_3} \mathbf{F}_jF_j > 0 \quad (4.5)$$

$$V_1' = \sum_{j=1}^{n_1} \mathbf{A}_jA_j - \sum_{j=n_1+1}^{p_1} \mathbf{A}_jA_j - \sum_{j=1}^{q_1} B_j\mathbf{B}_j + \sum_{j=1}^{n_4} \mathbf{G}_jG_j - \sum_{j=n_4+1}^{p_4} \mathbf{G}_jG_j + \sum_{j=1}^{m_4} \mathbf{H}_jH_j - \sum_{j=m_4+1}^{q_4} \mathbf{H}_jH_j > 0 \quad (4.6)$$

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The modified generalized I-function reduces to modified generalized H-function defined by Prasad and Prasad [10]. We have: $A_j = B_j = C_j = D_j = E_j = F_j = G_j = H_j = 1$ and

Corollary 3

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{\upsilon}}\ H(ZX_{a,b},Z'X_{c,d})\mathrm{d}x =$$

$$2\sqrt{\pi}z^{\alpha-\frac{1}{2}+\upsilon}\sum_{n'=0}^{\infty}\frac{a^{n'}}{z^{2n'}n'!}\left(\tfrac{1-\upsilon}{2}\right)^{n'}\left(\tfrac{-\upsilon}{2}\right)^{n'}H^{m_1,n_1+4:m_2,n_2:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_2,q_2:p_3,q_3:p_4,q_4}$$

$$\left(\begin{array}{c|c} 2^{-b}z^{a-b}Z & \mathbf{A}_1',\{(a_j;\alpha_j,A_j)\}_{1,p_1}:\{(c_i;\gamma_j,C_j)\}_{1,p_2}:\{(e_j;E_j)\}_{1,p_3},\{(g_j;G_j)\}_{1,p_4} \\ \vdots & \vdots \\ 2^{-d}z^{c-d}Z' & \{(\mathrm{b}_j;\beta_j,B_j)\}_{1,q_1},\mathbf{B}_1':\{(d_j;\delta_j,D_j)\}_{1,q_2}:\{(f_j;F_j)\}_{1,q_3},\{(h_j;H_j)\}_{1,q_4} \end{array}\right) \quad (4.7)$$

By respecting the conditions involving in the theorem with $(\mathbf{A_j}=\mathbf{B_j}=\mathbf{C_j}=\mathbf{D_j}=\mathbf{E_j}=\mathbf{F_j}=\mathbf{G_j}=\mathbf{H_j}=1)$ and

$|argz_1| < \frac{1}{2}U_1\pi, |argz_2| < \frac{1}{2}V_1\pi$, U_1 and V_1 are defined by the following formulas :

$$U_1=\sum_{j=1}^{n_1}\alpha_j-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=m_1+1}^{q_1}\beta_j+\sum_{j=1}^{n_2}\gamma_j-\sum_{j=n_2+1}^{p_2}\gamma_j+\sum_{j=1}^{m_2}\delta_j$$

$$-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j>0 \quad (4.8)$$

$$V_1=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j+\sum_{j=1}^{m_1}B_j-\sum_{j=m_1+1}^{q_1}B_j+\sum_{j=1}^{n_2}C_j-\sum_{j=n_2+1}^{p_2}C_j+\sum_{j=1}^{m_2}D_j$$

$$-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_4}G_j-\sum_{j=n_4+1}^{p_4}G_j+\sum_{j=1}^{m_4}H_j-\sum_{j=m_4+1}^{q_4}H_j>0 \quad (4.9)$$

where :

$$\mathbf{A}_1'=\left(\tfrac{\alpha-\upsilon}{2};\tfrac{b-a}{2},\tfrac{d-c}{2}\right),\left(1-\tfrac{\alpha}{2}-n';\tfrac{a}{2},\tfrac{c}{2}\right),\left(1-\tfrac{\upsilon+1}{2}-n';\tfrac{b}{2},\tfrac{d}{2}\right),(\text{-}\upsilon;b,d) \quad (4.10)$$

$$\mathbf{B}_1'=\left(-\tfrac{\upsilon}{2};\tfrac{b}{2},\tfrac{d}{2}\right),\ \left(1-\tfrac{\upsilon+1}{2};\tfrac{b}{2},\tfrac{d}{2}\right),(\text{-}\upsilon-n';b,d) \quad (4.11)$$

The generalized modified H-function reduces to the generalized modified of G-function of two variables defined by Agarwal [1], we suppose : a=b = c=d = 1 , then $A_1'=\left(\frac{\alpha-\upsilon}{2};0,0\right),\left(1-\frac{\alpha}{2}-n';\frac{1}{2},\frac{1}{2}\right),\left(1-\frac{\upsilon+1}{2}-n';\frac{1}{2},\frac{1}{2}\right)$, $(\text{-}\upsilon;1,1)$; $B_1'=\left(-\frac{\upsilon}{2};\frac{1}{2},\frac{1}{2}\right)$, $\left(1-\frac{\upsilon+1}{2};\frac{1}{2},\frac{1}{2}\right)$, $(\text{-}\upsilon-n';1,1)$ and we have the conditions :

$$(\alpha_j)_{1,p_1}=(A_j)_{1,p_1}=(\gamma_j)_{1,p_2}=(C_j)_{1,p_2}=(E_j)_{1,p_3}=(G_j)_{1,p_4}=1$$

$$=(\beta_j)_{1,q_1}=(B_j)_{1,q_1}=(\delta_j)_{1,q_2}=(D_j)_{1,q_2}=(F_j)_{1,q_3}=(H_j)_{1,q_4}$$

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, we have the result:

Corollary 4

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{v}} G(ZX, Z'X)\mathrm{d}x =$$

$$2\sqrt{\pi} z^{\alpha-\frac{1}{2}+v} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!} \left(\frac{1-v}{2}\right)^{n'} \left(\frac{-v}{2}\right)^{n'}$$

$$G^{m_1,n_1+4:m_2,n_2:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_2,q_2:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} \mathrm{Z} & \mathrm{A}'_1, (a_1)_{1,p_1} : (c_j)_{1,p_2} : (e_j)_{1,p_3}, (g_j)_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z'} & (\mathrm{b}_j)_{1,q_1}, B'_1 : (d_j)_{1,q_2} : (f_j)_{1,q_3}, (h_j)_{1,q_4} \end{array}\right) \tag{4.12}$$

By considering the conditions verified by the corollary 3, about the modified generalized H-function of two variables and $(\alpha_j)_{1,p_1} = (A_j)_{1,p_1} = (\gamma_j)_{1,p_2} = (C_j)_{1,p_2} = (E_j)_{1,p_3} = (G_j)_{1,p_4} = 1$, where

$$X = \frac{x}{(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})} \tag{4.13}$$

$|argZ| < \frac{1}{2}U_1\pi, |argZ'| < \frac{1}{2}V_1\pi$, U_1, V_1 are defined by the following formulas :

$$\mathrm{U}_1 = \left[m_1+n_1+m_2+n_2+m_3+n_3-\tfrac{1}{2}(p_1+q_1+p_2+q_2+p_3+q_3)\right] \tag{4.14}$$

and

$$\mathrm{V}_1 = \left[m_1+n_1+m_2+n_2+m_4+n_4-\tfrac{1}{2}(p_1+q_1+p_2+q_2+p_4+q_4)\right] \tag{4.15}$$

In the following, we suppose $m_1 = m_2 = n_2 = p_2 = q_2 = 0$, we consider the H-function of two variables defined by K.C. Gupta, and P.K. Mittal [6] or the corollary 2 (I-function of two variables studied by Kumari et al. [8] and let: (Aj = Cj = Dj = Fj = Gj = Hj = 1), we obtain the relation:

Corollary 5

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^{v}} H(ZX_{a,b}, Z'X_{c,d})\mathrm{d}x =$$

$$2\sqrt{\pi} z^{\alpha-\frac{1}{2}+v} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!} \left(\frac{1-v}{2}\right)^{n'} \left(\frac{-v}{2}\right)^{n'}$$

$$H^{0,n_1+4:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} 2^{-b}z^{a-b}Z & \mathbf{A}'_1, \{(a_j;\alpha_j,A_j)\}_{1,p_1} : \{(e_j;E_j)\}_{1,p_3}, \{(g_j;G_j)\}_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{-d}z^{c-d}Z' & \{(\mathrm{b}_j;\beta_j,B_j)\}_{1,q_1}, \mathbf{B}'_1 : \{(f_j;F_j)\}_{1,q_3}, \{(h_j;H_j)\}_{1,q_4} \end{array}\right) \tag{4.16}$$

By using the conditions mentionned in the corollary 2 with $\mathrm{m}_1 = m_2 = n_2 = p_2 = q_2 = 0$, and we have :

$|argz_1| < \frac{1}{2}U''_1\pi, |argz_2| < \frac{1}{2}V''_1\pi$, U''_1 where V''_1.

$$U_1'' = \sum_{j=1}^{n_1} \alpha_j - \sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j > 0 \quad (4.17)$$

$$V_1'' = \sum_{j=1}^{n_1} A_j - \sum_{j=n_1+1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{n_4} G_j - \sum_{j=n_4+1}^{p_4} G_j + \sum_{j=1}^{m_4} H_j - \sum_{j=m_4+1}^{q_4} H_j > 0 \quad (4.18)$$

The quantities $\mathbf{A}_1'$ and $\mathbf{B}_1'$ are defined respectively by the formulas (4.10) and (4.11).

Taking conditions:

$m_1 = 0; (\alpha_j)_{1,p_1} = (A_j)_{1,p_1} = (E_j)_{1,p_3} = (G_j)_{1,p_4} = 1 = (\beta_j)_{1,q_1} = (B_j)_{1,q_1} = (F_j)_{1,q_3} = (H_j)_{1,q_4}$, we have the result with The G-function of two variables defined by Agarwal [1].

Corollary 6

$$\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2y^2+z^2)}(\sqrt{x^2+y^2}+\sqrt{x^2y^2+z^2})^v} G(ZX, Z'X)\mathrm{d}x =$$

$$2\sqrt{\pi} z^{\alpha-\frac{1}{2}+v} \sum_{n'=0}^{\infty} \frac{a^{n'}}{z^{2n'} n'!} \left(\frac{1-v}{2}\right)^{n'} \left(\frac{-v}{2}\right)^{n'}$$

$$G^{0,n_1+4:m_3,n_3:m_4,n_4}_{p_1+4,q_1+3:p_3,q_3:p_4,q_4}\left(\begin{array}{c|c} \mathrm{Z} & \mathrm{A}_1', (a_1)_{1,p_1} : (e_j)_{1,p_3}, (g_j)_{1,p_4} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z'} & (\mathrm{b}_i)_{1,q_1}, B_1' : (f_j)_{1,q_3}, (h_j)_{1,q_4} \end{array}\right) \quad (4.19)$$

under the same notations and conditions that the above corollary and the following conditions are respected.
$m_1 = 0; (\alpha_j)_{1,p_1} = (A_j)_{1,p_1} = (E_j)_{1,p_3} = (G_j)_{1,p_4} = 1 = (\beta_j)_{1,q_1} = (B_j)_{1,q_1} = (F_j)_{1,q_3} = (H_j)_{1,q_4}$, and

$|argZ| < \frac{1}{2}U_1\pi, |argZ'| < \frac{1}{2}V_1\pi$, U_1 and V_1 are defined by the following formulas :

$$\mathrm{U}_1 = \left[n_1 + m_3 + n_3 - \tfrac{1}{2}(p_1 + q_1 + p_3 + q_3)\right] \quad (4.20)$$

and

$$\mathrm{V}_1 = \left[n_1 + m_4 + n_4 - \tfrac{1}{2}(p_1 + q_1 + p_4 + q_4)\right] \quad (4.21)$$

Remarks

We have the same generalized finite integral with the modified generalized of I-function of two variables defined by Kumari et al. [8], see Singh and Kumar for more details [15] and the special cases, the I-function defined by Saxena [13], the I-function defined by Rathie [12], the Fox's H-function, We have the same generalized multiple finite integrals with the incomplete aleph-function defined by Bansal et al. [3], the incomplete I-function studied by Bansal and Kumar. [2] and the incomplete Fox's H-function given by Bansal et al. [4], the Psi function defined by Pragathi et al. [9].

We have the same generalized finite integral with the generalized modified of the aleph-function of two variables defined by Kumar [7], Sharma [14].

V. CONCLUSION

The importance of our all the results lies in their manifold generality. First by specializing the various parameters as well as variable of the generalized modified I-function of two variables, we obtain a large number of results involving remarkably wide variety of useful special functions (or product of such special functions) which are expressible in terms of I-function of two variables or one variable defined by Rathie [12], H-function of two or one variables , Meijer's G-function of two or one variables and hypergeometric function of two or one variables . Secondly, by specializing the parameters of this unified finite integral, we can get a large number of integrals involving the

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modified generalized I-functions of two variables and the others functions seen in this document.

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